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1997 J. Phys. A: Math. Gen. 30 4341

(http://iopscience.iop.org/0305-4470/30/12/021)

J. Phys. A: Math. Gen. **30** (1997) 4341–4364. Printed in the UK PII: S0305-4470(97)81094-3

Quantum integrability of the generalized elliptic Ruijsenaars models

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Received 20 January 1997

Abstract. The quantum integrability of the generalized elliptic Ruijsenaars models is shown. These models are mathematically related to the Macdonald operator and the Macdonald–Koornwinder operator, which appeared in the q-orthogonal polynomial theories. We construct these integrable families by using the Yang–Baxter equation and the reflection equation.

1. Introduction

The Ruijsenaars model [1] was introduced as a relativistic quantum N-body system in one dimension, whose Hamiltonian was defined as

$$\begin{aligned}\mathcal{H}_{\mathrm{A}} = \sum_{j=1}^{N} \left(\prod_{\substack{k=1\\k\neq j}}^{N} \sqrt{\frac{\vartheta_1(z_{jk}-\mu)}{\vartheta_1(z_{jk})}} \right) \exp\left(\beta \frac{\partial}{\partial z_j}\right) \left(\prod_{\substack{k=1\\k\neq j}}^{N} \sqrt{\frac{\vartheta_1(z_{kj}-\mu)}{\vartheta_1(z_{kj})}} \right) \\ + \sum_{j=1}^{N} \left(\prod_{\substack{k=1\\k\neq j}}^{N} \sqrt{\frac{\vartheta_1(z_{kj}-\mu)}{\vartheta_1(z_{kj})}} \right) \exp\left(-\beta \frac{\partial}{\partial z_j}\right) \left(\prod_{\substack{k=1\\k\neq j}}^{N} \sqrt{\frac{\vartheta_1(z_{jk}-\mu)}{\vartheta_1(z_{jk})}} \right).\end{aligned} \tag{1.1}$$

Here β and μ are constants, and z_{jk} denotes $z_j - z_k$. The function $\vartheta_r(z)$ is the Jacobi theta function (see the appendix for detail). This model is integrable and reduces to the Calogero–Sutherland–Moser (CSM) model in the non-relativistic limit. The Hamiltonian $\mathcal{H}_{\mathrm{A}}$ is invariant under $z_j \leftrightarrow z_k$, and called the A-type model. A D-type analogue of the model exists, i.e. invariant under $z_j \leftrightarrow -z_j$ and $z_j \leftrightarrow z_k$ [2, 3]. In this context we propose a new integrable Hamiltonian of type D, defined by

$$\begin{aligned}\mathcal{H}_{\mathrm{D}} = \sum_{j=1}^{N} \Psi_j(z)^{1/2} \exp\left(2\beta \frac{\partial}{\partial z_j}\right) \Psi_j(-z)^{1/2} \\ + \sum_{j=1}^{N} \Psi_j(-z)^{1/2} \exp\left(-2\beta \frac{\partial}{\partial z_j}\right) \Psi_j(z)^{1/2} + \Psi_0(z)\end{aligned} \tag{1.2}$$

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where functions $\Psi_j(z)$ and $\Psi_0(z)$ are

$$\Psi_j(z) = \left(\prod_{\substack{k=1 \\ k\neq j}}^{N} \frac{\vartheta_1(z_{jk}-\mu)}{\vartheta_1(z_{jk})} \frac{\vartheta_1(z_j+z_k-\mu)}{\vartheta_1(z_j+z_k)} \right) \left(\prod_{r=0}^{3} \frac{\vartheta_{r+1}(z_j-\nu_r)}{\vartheta_{r+1}(z_j)} \frac{\vartheta_{r+1}(z_j+\beta-\bar{\nu}_r)}{\vartheta_{r+1}(z_j+\beta)} \right) \tag{1.3a}$$

$$\begin{aligned}\Psi_0(z) = -\sum_{p=0}^{3} \left(\frac{\pi}{\vartheta_1'(0)}\right)^2 \frac{2}{\vartheta_1(-\mu)\vartheta_1(-2\beta-\mu)} \left(\prod_{r=0}^{3} \vartheta_{r+1}(-\beta-\nu_{\pi_p r})\vartheta_{r+1}(-\bar{\nu}_{\pi_p r}) \right) \\ \times \left(\prod_{j=1}^{N} \frac{\vartheta_{p+1}(z_j-\beta-\mu)}{\vartheta_{p+1}(z_j-\beta)} \frac{\vartheta_{p+1}(-z_j-\beta-\mu)}{\vartheta_{p+1}(-z_j-\beta)} \right).\end{aligned} \tag{1.3b}$$

Here π_p denotes permutation; $\pi_0 = \mathbb{1}$, $\pi_1 = (01)(23)$, $\pi_2 = (02)(13)$, and $\pi_3 = (03)(12)$. We note that this model contains 10 parameters, β, μ, ν_r and $\bar{\nu}_r$ ($r = 0, 1, 2, 3$). The potentials of these models are indeed elliptic functions, but only the trigonometric cases have received much attention since the operators $\mathcal{H}_{\mathrm{A}}$ and $\mathcal{H}_{\mathrm{D}}$ are respectively related to the Macdonald operator [4, 5] and the Macdonald–Koornwinder operator [6] in the q-orthogonal polynomial theory. Recently it has been revealed that they can be treated by means of the operator-valued solutions [7–10] of the Yang–Baxter equation (YBE) and the reflection equation (RE), and that the affine Hecke algebra plays a crucial role in the Macdonald theory [11–13]. Although some attempts have been made for the quantum elliptic Ruijsenaars model and the quantum elliptic CSM model [14–17], the systematic studies of these models are still lacking in contrast to the trigonometric case or the classical case [18, 19]. In view of the elliptic Ruijsenaars model of type D, no technical tools were known to work well, and the integrability was only conjectured by using the direct construction of all the conserved operators and the direct calculation of the commutativity [2, 3].

In this paper, we shall give a new construction of the Ruijsenaars model by using the operator-valued solutions of the YBE and the RE. This method is quite different from the existing ones, and works well even in the elliptic case. It can be clarified by pictorial interpretation that the models are integrable and that commuting operators exist. In addition, for the A-type model, all the conserved elliptic operators can be computed, and the duality can easily be seen.

This paper is organized as follows. In section 2, we briefly review the operator-valued solutions of the YBE and the RE following [10]. They respectively include one and four arbitrary parameters besides a scaling factor. Based on these solutions, we prove the integrability of the elliptic Ruijsenaars model of type A in section 3. We define a set of mutually commuting difference operators and show that they coincide with the elliptic Macdonald operators. The integrability of the Ruijsenaars model follows from the gauge transformation of the elliptic Macdonald operator. The duality of the Macdonald operators is also discussed. We shall apply the same scheme to the generalized elliptic Ruijsenaars model. In section 4, we construct a family of commuting operators related to the RE. Our integrable difference operators of type D can be regarded as a 10 parameter generalization of the elliptic Macdonald–Koornwinder operators. In fact, we show that our operator includes the previously known D-type operator studied in [2, 3]. The final section is devoted to concluding remarks.

2. Yang-Baxter equation and reflection equation

We consider solutions of the YBE and the RE [20, 21], respectively defined by

$$R^{12}(\xi_{12})R^{13}(\xi_{13})R^{23}(\xi_{23}) = R^{23}(\xi_{23})R^{13}(\xi_{13})R^{12}(\xi_{12}) \tag{2.1}$$

$$R^{12}(\xi_{12})\overset{1}{K}(\xi_1)R^{21}(\xi_1+\xi_2)\overset{2}{K}(\xi_2) = \overset{2}{K}(\xi_2)R^{12}(\xi_1+\xi_2)\overset{1}{K}(\xi_1)R^{21}(\xi_{12}). \tag{2.2}$$

Here ξ_j is called the spectral parameter, and ξ_{jk} denotes $\xi_{jk} = \xi_j - \xi_k$. The operator $R^{jk}(\xi)$ acts non-trivially on the jth and kth spaces, and $\overset{j}{K}(\xi)$ acts as $K(\xi)$ only on the jth space.

Following an idea from [7, 8], we regard $R(\xi)$ and $K(\xi)$ as operators acting on functional spaces. In this sense such R- and K-operators may be viewed as an infinite-dimensional representation for solutions of the YBE and the RE. Here, to study operator-valued solution of the YBE (2.1) and the RE (2.2), we set the R- and K-operators acting on functional space as [9]

$$R^{jk}(\xi) = A(z_{jk}) - B(\xi, z_{jk})\hat{s}_{jk} \tag{2.3}$$

$$\overset{j}{K}(\xi) = G(\xi, z_j) - H(z_j)\hat{t}_j \tag{2.4}$$

where functions $A(z)$, $B(\xi, z)$, $G(\xi, z)$ and $H(z)$ are to be determined. Operators $\hat{s}_{jk}$ and $\hat{t}_j$ are respectively an exchange operator and a reflection operator, satisfying relations,

$$\begin{aligned} &\hat{s}_{jk}^2 = \mathbb{1} \qquad \hat{s}_{jk}\hat{s}_{kl} = \hat{s}_{kl}\hat{s}_{lj} = \hat{s}_{lj}\hat{s}_{jk} \qquad \hat{s}_{jk}z_j = z_k\hat{s}_{jk} \\ &\hat{t}_j^2 = \mathbb{1} \qquad \hat{t}_j\hat{s}_{jk} = \hat{s}_{jk}\hat{t}_k \qquad \hat{t}_j z_j = -z_j\hat{t}_j. \end{aligned}$$

We suppose that the functions $B(\xi, z)$ and $G(\xi, z)$ are odd,

$$B(\xi, z) = -B(-\xi, -z) \tag{2.5}$$

$$G(\xi, z) = -G(-\xi, -z). \tag{2.6}$$

As was proved in [13], we have functional equations as conditions to satisfy both the YBE and the RE.

Proposition 2.1. The R-operator (2.3) satisfies the YBE (2.1) when functions $A(z)$ and $B(\xi, z)$ satisfy the following functional equations;

$$A(z_1)A(-z_1) - B(\xi, z_1)B(\xi, -z_1) = A(z_2)A(-z_2) - B(\xi, z_2)B(\xi, -z_2) \equiv c(\xi) \tag{2.7a}$$

$$B(\xi_1, z_1)B(-\xi_2, z_{12}) = B(\xi_{12}, z_{12})B(\xi_1, z_2) + B(-\xi_2, -z_2)B(\xi_{12}, z_1). \tag{2.7b}$$

Proposition 2.2. The K-operator (2.4) and the R-operator (2.3) satisfy the RE (2.2) if the following functional equations are fulfilled;

$$\begin{aligned} G(\xi_1, z_1)B(\xi_1+\xi_2, z_{21}) + G(\xi_1, z_2)B(\xi_{12}, z_{12}) + G(\xi_2, -z_1)B(\xi_1+\xi_2, z_1+z_2) \\ = G(\xi_2, z_2)B(\xi_{12}, z_1+z_2) \end{aligned} \tag{2.8a}$$

$$H(z_1)H(-z_1) - G(\xi, z_1)G(\xi, -z_1) = H(z_2)H(-z_2) - G(\xi, z_2)G(\xi, -z_2) \equiv d(\xi). \tag{2.8b}$$

As explicit solutions of the functional equations (2.7) and (2.8), we have the following theorems [8, 10].

Theorem 2.3. The elliptic R-operator, defined by

$$R^{jk}(\xi) = \sigma_\mu(z_{jk}) - \sigma_\xi(z_{jk})\hat{s}_{jk} \tag{2.9}$$

with arbitrary constant μ, satisfies the YBE (2.1).

Theorem 2.4. With the elliptic R-operator (2.9), the elliptic K-operator defined by

$$K(\xi) = \sum_{r=0}^{3} g_r \sigma_{2\xi}^r(z) - \sum_{r=0}^{3} g_r \sigma_{2\nu}^r(z)\hat{t} \tag{2.10}$$

satisfies the RE (2.2). Here parameters ν, g_r $(r = 0, 1, 2, 3)$ are arbitrary.

See [10, 13] for the proof of these propositions and theorems. We note that the R-operator (2.9) and the K-operator (2.10) satisfy the unitarity conditions,

$$R^{12}(\xi)R^{21}(-\xi) = c(\xi)\mathbb{1} \tag{2.11}$$

$$K(\xi)K(-\xi) = d(\xi)\mathbb{1}. \tag{2.12}$$

Here function $c(\xi)$ is given as $c(\xi) = \wp(\mu) - \wp(\xi)$. We do not use the explicit form $d(\xi)$ in the following.

3. Elliptic Ruijsenaars model of type A

The difference analogue of the elliptic CSM model was introduced by Ruijsenaars [1] for A type, whose Hamiltonian is given by

$$\mathcal{H}_{\mathrm{A}} = \sum_{j=1}^{N} \left(\prod_{\substack{k=1 \\ k\neq j}}^{N} \sqrt{\frac{\vartheta_1(z_{jk}-\mu)}{\vartheta_1(z_{jk})}} \right) \exp\left(\beta \frac{\partial}{\partial z_j}\right) \left(\prod_{\substack{k=1 \\ k\neq j}}^{N} \sqrt{\frac{\vartheta_1(z_{kj}-\mu)}{\vartheta_1(z_{kj})}} \right)$$
$$+ \sum_{j=1}^{N} \left(\prod_{\substack{k=1 \\ k\neq j}}^{N} \sqrt{\frac{\vartheta_1(z_{kj}-\mu)}{\vartheta_1(z_{kj})}} \right) \exp\left(-\beta \frac{\partial}{\partial z_j}\right) \left(\prod_{\substack{k=1 \\ k\neq j}}^{N} \sqrt{\frac{\vartheta_1(z_{jk}-\mu)}{\vartheta_1(z_{jk})}} \right) \tag{3.1}$$

where μ, β and τ are pure imaginary so that the Hamiltonian is Hermitian. We suppose that the Ruijsenaars model consists of one-component boson, and that the Hamiltonian $\mathcal{H}_{\mathrm{A}}$ (3.1) acts on the symmetric space in coordinates $z_1, z_2, \ldots, z_N$. Thus, hereafter we restrict our discussions on the symmetric space, i.e. we replace exchange operators $\hat{s}_{jk}$ by $\mathbb{1}$ when they are moved to the rightmost of the expression.

For later convenience, we introduce a gauge-transformed Hamiltonian as [17]

$$\tilde{\mathcal{H}}_{\mathrm{A}} = \Delta_{\mathrm{A}}^{-1/2} \mathcal{H}_{\mathrm{A}} \Delta_{\mathrm{A}}^{1/2}$$
$$\equiv \mathcal{M}_1(\mu, \beta) + \mathcal{M}_1(-\mu, -\beta). \tag{3.2}$$

The function of the gauge transformation $\Delta_{\mathrm{A}}^{1/2}$ is given by

$$\Delta_{\mathrm{A}} = \prod_{\substack{j,k=1 \\ j\neq k}}^{N} \frac{(pv_j/v_k;\, p, q)_\infty}{(pw^{-1}v_j/v_k;\, p, q)_\infty} \frac{(qwv_k/v_j;\, p, q)_\infty}{(qv_k/v_j;\, p, q)_\infty} \tag{3.3}$$

where we set parameters as $v_j = \mathrm{e}^{2\pi \mathrm{i} z_j}$, $w = \mathrm{e}^{2\pi\mathrm{i}\mu}$, $p = \mathrm{e}^{2\pi\mathrm{i}\tau}$, and $q = \mathrm{e}^{2\pi\mathrm{i}\beta}$. The double infinite product $(x;\, p, q)_\infty$ denotes

$$(x;\, p, q)_\infty = \prod_{m=0}^{\infty}\prod_{n=0}^{\infty}(1 - xp^m q^n). \tag{3.4}$$

To ensure the condition $|p| < 1$ and $|q| < 1$ for convergence, we set $\tau \in \mathrm{i}\mathbb{R}^+$ and $\beta \in \mathrm{i}\mathbb{R}^+$. The difference operator $\mathcal{M}_1(\mu, \beta)$ is computed as

$$\mathcal{M}_1(\mu, \beta) = \sum_{1\leqslant j\leqslant N} \left(\prod_{\substack{k=1 \\ k\neq j}}^{N} \frac{\vartheta_1(z_{jk}-\mu)}{\vartheta_1(z_{jk})} \right) T_j(\beta) \tag{3.5}$$

with the shift operator $T_j(\beta) = \exp(\beta \frac{\partial}{\partial z_j})$. In recent mathematical terminology, the trigonometric limit of the operator (3.5) is called the (A-type) Macdonald operator, and its eigenfunctions are known as the Macdonald symmetric polynomials [5].

We shall show, in this section, that the Ruijsenaars model, or the elliptic Macdonald operator (3.2), is integrable, and that N independent commutative operators exist including the Hamiltonian (3.2);

- the higher-order Macdonald difference operators, which are defined by

$$\mathcal{M}_n(\mu, \beta) = \sum_{\substack{I \subset \{1,\dots,N\} \\ |I|=n}} \left(\prod_{\substack{j \in I \\ k \in I^c}} \frac{\vartheta_1(z_{jk} - \mu)}{\vartheta_1(z_{jk})} \right) T_I(\beta) \tag{3.6}$$

form a commuting family. Here we set $T_I(\beta) = \prod_{j \in I} T_j(\beta)$.

- The Ruijsenaars model has a 'duality'; the difference operators $\mathcal{M}_n(-\mu, -\beta)$ can be written in terms of $\mathcal{M}_1(\mu, \beta), \dots, \mathcal{M}_N(\mu, \beta)$.

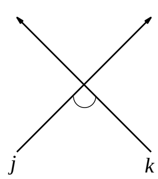
We shall construct a set of the elliptic Macdonald operators (3.6) from the elliptic R-operator (2.9), and prove that they constitute a mutually commuting family. For brevity, we omit μ and β in $\mathcal{M}_n(\mu, \beta)$ and $T_j(\beta)$, and denote $\mathcal{M}_n = \mathcal{M}_n(\mu, \beta)$, $T_j = T_j(\beta)$ unless explicitly indicated.

First we define a set of difference operators $\mathcal{D}_n(\xi) = \mathcal{D}_n(\xi_1, \dots, \xi_N)$ as

$$\mathcal{D}_n(\xi) = \prod_{m=N-n+1}^{N} \left(\prod_{k=1}^{\overleftarrow{N-n}} R^{mk}(\xi_{mk}) \right) T_m. \tag{3.7}$$

To clarify the structure of our difference operators, we assign a figure for each operator;

R^{jk} . / =

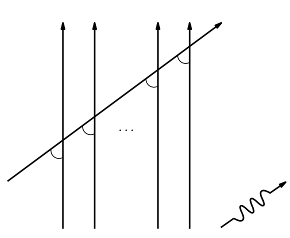


(3.8)

T_j . / = j

Using these figures, the difference operators $\mathcal{D}_n(\xi)$ are depicted as follows. Note that both ends of the figures are supposed to be connected.

$\mathcal{D}_1$. / =



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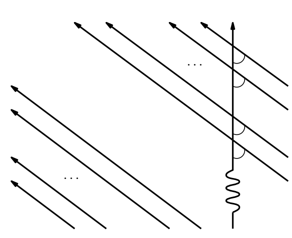
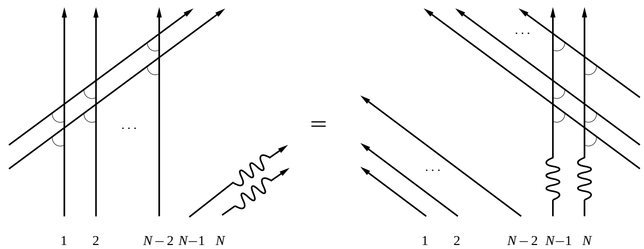


(3.9a)

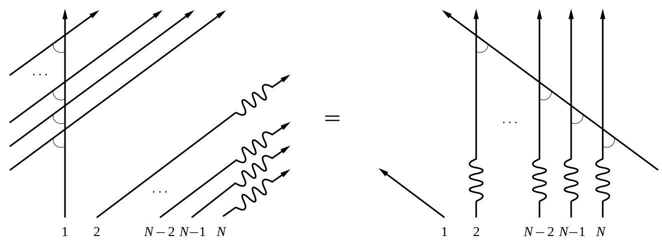
$\mathcal{D}_2 \cdot / =$



(3.9*b*)

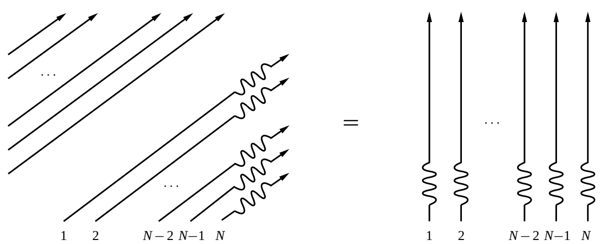
$\vdots$

$\mathcal{D}_{N-1} \cdot / =$



(3.9*c*)

$\mathcal{D}_N \cdot / =$



(3.9*d*)

One sees the similarity between $\mathcal{D}_n(\xi)$ and $\mathcal{D}_{N-n}(\xi)$ from above figures. In fact this leads us to an important property of the Macdonald operators, 'duality'.

First we shall prove that difference operators $\mathcal{D}_n(\xi)$ constitute a mutually commuting family. Therein we use the following proposition.

Proposition 3.1. The elliptic R-operator (2.9) commutes with $T_j T_k$;

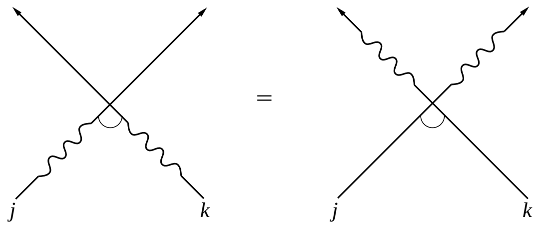
$$R^{jk}(\xi) T_j T_k = T_j T_k R^{jk}(\xi). \tag{3.10}$$

Proof. This relation is directly proved from the fact that the R-operator (2.9) depends only on the difference of coordinates. □

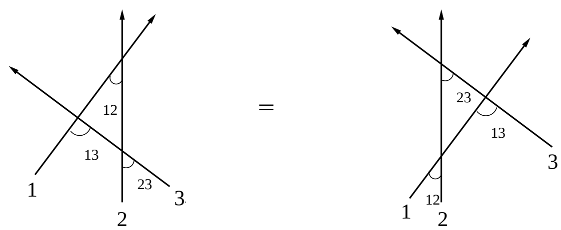
Theorem 3.2. The difference operators $\mathcal{D}_n(\xi)$ defined in (3.7) are integrable,

$$[\mathcal{D}_l(\xi), \mathcal{D}_m(\xi)] = 0 \qquad \text{for } 1 \leqslant l,\ m \leqslant N. \tag{3.11}$$

Proof. We prove this theorem by the extended 'railway argument'. To this end, we depict the relation (3.10) and the YBE (2.1) as,



$$R^{jk}(\cdot)T_j\,T_k \quad = \quad T_j\,T_k\,R^{jk}(\cdot) \tag{3.10$'$}$$



$$R^{12}(\cdot_{12})R^{13}(\cdot_{13})R^{23}(\cdot_{23}) \quad = \quad R^{23}(\cdot_{23})R^{13}(\cdot_{13})R^{12}(\cdot_{12}). \tag{2.1$'$}$$

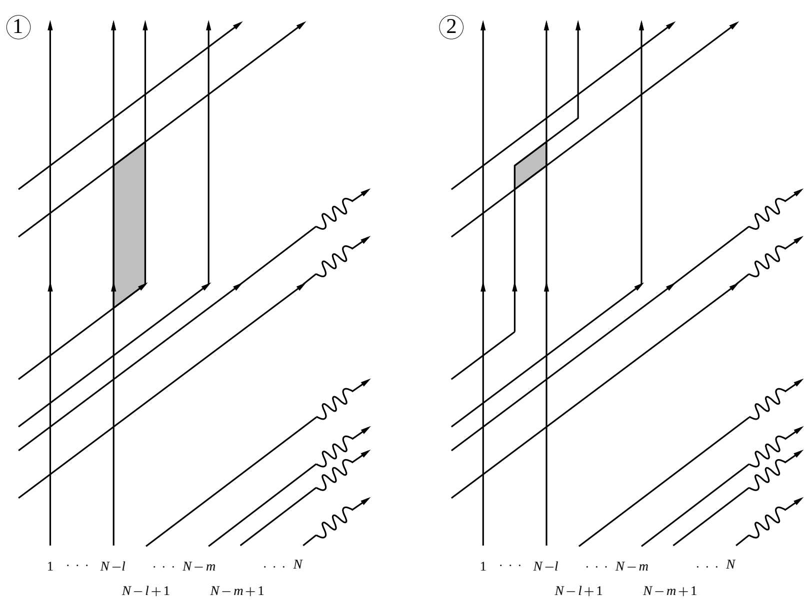
The proof consists of four steps;

(1) we depict the product $\mathcal{D}_m(\xi)\mathcal{D}_l(\xi)$ for $l > m$ (the left of (3.12*a*)).

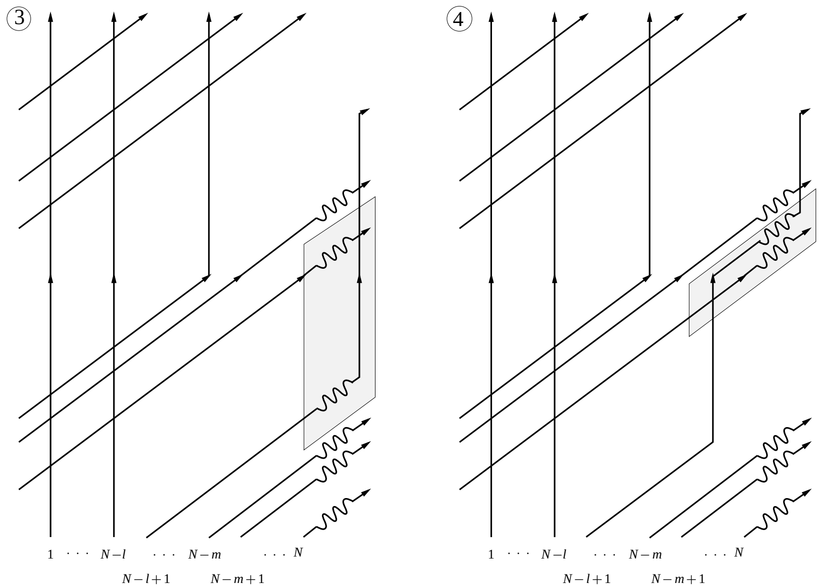
(2) Repeatedly using the YBE (2.1), we bring the $(N-l)$th line to the top (the right of (3.12*a*)).

(3) Repeatedly using relation (3.10), we finish raising the $(N-l)$th line as depicted in (3.12*b*) and the left of (3.12*c*).

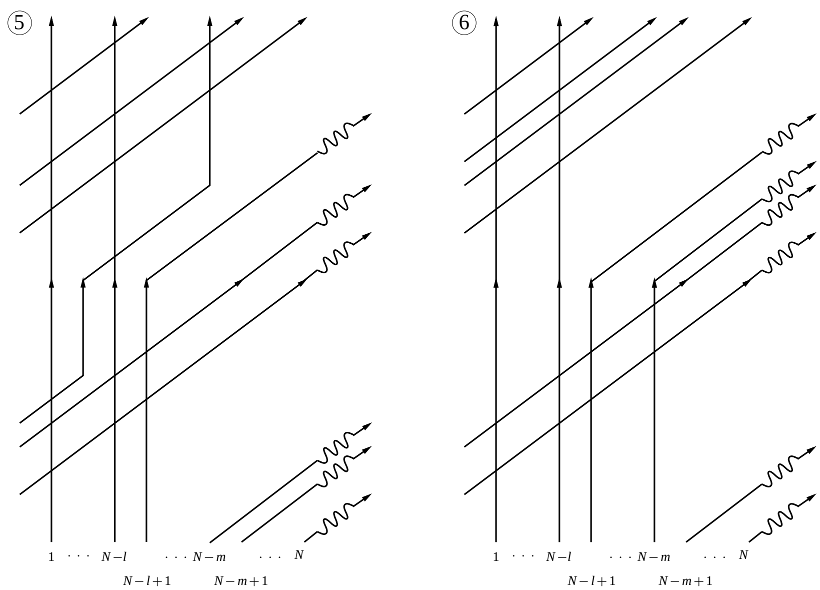
(4) Successive application of steps 2 and 3 gives us an equation (the right of (3.12*c*)), which represents the product $\mathcal{D}_l(\xi)\mathcal{D}_m(\xi)$.



(3.12*a*)



(3.12b)



(3.12c)

These steps complete the proof of the commutativity, $\mathcal{D}_m(\xi)\mathcal{D}_l(\xi) = \mathcal{D}_l(\xi)\mathcal{D}_m(\xi)$. □

We shall compute the explicit forms of the integrable difference operators $\mathcal{D}_n(\xi)$.

Theorem 3.3. The integrable difference operators $\mathcal{D}_n(\xi)$ defined in (3.7) agree with the elliptic Macdonald operators (3.6) up to constants;

$$\mathcal{D}_n = \alpha_n \mathcal{M}_n \qquad \text{for } 1 \leqslant n \leqslant N \tag{3.13}$$

where operators $\mathcal{D}_n$ denote $\mathcal{D}_n(\xi)$ (3.7) with spectral parameters $\xi_{mk} = (k-m)\mu$. Constant α_n is defined as $\alpha_n = (\vartheta_1'(0)/\vartheta_1(-\mu))^{(N-n)n}$.

In the following, we shall prove theorem 3.3, which is a main theorem of this section. We point out that the difference operator $\mathcal{D}_n$ is written as

$$\mathcal{D}_n = \prod_{m=N-n+1}^{N} D_{N-n}^m \tag{3.14}$$

with D_j^m defined by

$$D_j^m = \left(\overleftarrow{\prod_{k=1}^{j}} R^{mk}(\xi_{mk}) \right) T_m. \tag{3.15}$$

When we substitute the explicit form of the elliptic R-operator (2.9) into the difference operator D_j^m, we find that the operator D_j^m is written as

$$D_j^m = \sum_{l=1}^{j} F(m, j; l, l) + \left(\prod_{k=1}^{j} \sigma_\mu(z_{mk}) \right) T_m \tag{3.16}$$

where $F(m, j; l, q)$ is defined by

$$F(m, j; l, q) = \left(\overleftarrow{\prod_{k=q+1}^{j}} (\sigma_\mu(z_{mk}) + \sigma_{(m-k)\mu}(z_{km})\hat{s}_{mk}) \right) \sigma_{(m-q)\mu}(z_{lm}) \left(\prod_{\substack{k=1 \\ k \neq l}}^{q} \sigma_\mu(z_{lk}) \right) T_l \hat{s}_{ml}. \tag{3.17}$$

As a property of an operator $F(m, j; l, q)$, we have the following identity;

Lemma 3.4. When we suppose that operators $F(m, j; l, q)$ act on the symmetric space of $z_1, \ldots, z_j$, we have

$$F(m, j; l, l) = F(m, j; l, j). \tag{3.18}$$

Proof. This lemma is proved by showing an identity, $F(m, j; l, q) = F(m, j; l, q+1)$ for arbitrary q;

$$\begin{aligned}
F(m, j; l, q) &= \left(\overleftarrow{\prod_{k=q+2}^{j}} (\sigma_\mu(z_{mk}) + \sigma_{(m-k)\mu}(z_{km})\hat{s}_{mk}) \right) \\
&\quad \times (\sigma_\mu(z_{mq+1})\sigma_{(m-q)\mu}(z_{lm}) + \sigma_{(m-q-1)\mu}(z_{q+1m})\sigma_{(m-q)\mu}(z_{lq+1})) \\
&\quad \times \left(\prod_{\substack{k=1 \\ k \neq l}}^{q} \sigma_\mu(z_{lk}) \right) T_l \hat{s}_{ml} \\
&= \left(\overleftarrow{\prod_{k=q+2}^{j}} (\sigma_\mu(z_{mk}) + \sigma_{(m-k)\mu}(z_{km})\hat{s}_{mk}) \right) \\
&\quad \times \sigma_{(m-q-1)\mu}(z_{lm})\sigma_\mu(z_{lq+1}) \left(\prod_{\substack{k=1 \\ k \neq l}}^{q} \sigma_\mu(z_{lk}) \right) T_l \hat{s}_{ml} \\
&= F(m, j; l, q+1)
\end{aligned}$$

where, in the second equality, we have used the addition formula (A.6) and an identity, $\hat{s}_{mk}\hat{s}_{ml} = \hat{s}_{ml}\hat{s}_{kl}$. □

Note that operator D_j^m (3.15) is symmetric in $z_1, \ldots, z_j$ because operator $F(m, j; l, j)$ is written as

$$F(m, j; l, j) = \sigma_{(m-j)\mu}(z_{lm}) \Bigg(\prod_{\substack{k=1 \\ k \neq l}}^{j} \sigma_\mu(z_{lk}) \Bigg) T_l \hat{s}_{ml}. \tag{3.19}$$

Thus from lemma 3.4, we can replace all $F(m, N-n; l, l)$ in $\mathcal{D}_n$ (3.14) by $F(m, N-n; l, N-n)$ which is given as (3.19). By definition, the lowest-order difference operator is calculated as

$$\begin{aligned}
\mathcal{D}_1 = D_{N-1}^N &= \sum_{l=1}^{N-1} F(N, N-1; l, l) + \Bigg(\prod_{k=1}^{N-1} \sigma_\mu(z_{Nk}) \Bigg) T_N \\
&= \sum_{l=1}^{N-1} F(N, N-1; l, N-1) + \Bigg(\prod_{k=1}^{N-1} \sigma_\mu(z_{Nk}) \Bigg) T_N \\
&= \sum_{l=1}^{N-1} \sigma_\mu(z_{lN}) \Bigg(\prod_{\substack{k=1 \\ k \neq l}}^{N-1} \sigma_\mu(z_{lk}) \Bigg) T_l \hat{s}_{Nl} + \Bigg(\prod_{k=1}^{N-1} \sigma_\mu(z_{Nk}) \Bigg) T_N \\
&= \sum_{j=1}^{N} \Bigg(\prod_{\substack{k=1 \\ k \neq j}}^{N} \frac{\vartheta_1'(0)\vartheta_1(z_{jk}-\mu)}{\vartheta_1(z_{jk})\vartheta_1(-\mu)} \Bigg) T_j = \alpha_1 \mathcal{M}_1
\end{aligned} \tag{3.20}$$

where we have supposed that the functional space is symmetric.

To compute the explicit forms of the higher difference operators $\mathcal{D}_n$ for $n > 1$, we use the following lemma.

Lemma 3.5. Let D_j^m be defined by (3.15). The following identity holds;

$$\prod_{m=N-n+1}^{N-n+q} D_{N-n}^m = \sum_{\substack{I \subset \{1,\ldots,N-n+q\} \\ |I|=q}} \Bigg(\prod_{\substack{i \in I \\ k \in I^c}} \sigma_\mu(z_{ik}) \Bigg) T_I \Bigg(\prod_{j \in I \cap \{1,\ldots,N-n\}} \hat{s}_{j\phi_I(j)} \Bigg) \tag{3.21}$$

where ϕ_I is some one-to-one mapping from $I \cap \{1, \ldots, N-n\}$ to $I^c \cap \{N-n+1, \ldots, N-n+q\}$.

Proof. We prove this lemma by mathematical induction for q. First we set $A_q = \{1, \ldots, N-n+q\}$. For the $q = 1$ case, we obtain

$$\begin{aligned}
D_{N-n}^{N-n+1} &= \sum_{i=1}^{N-n} \Bigg(\prod_{\substack{k=1 \\ k \neq i}}^{N-n+1} \sigma_\mu(z_{ik}) \Bigg) T_i \hat{s}_{N-n+1i} + \Bigg(\prod_{k=1}^{N-n} \sigma_\mu(z_{N-n+1k}) \Bigg) T_{N-n+1} \\
&= \sum_{\substack{I \subset A_1 \\ |I|=1}} \Bigg(\prod_{\substack{i \in I \\ k \in I^c}} \sigma_\mu(z_{ik}) \Bigg) T_I \Bigg(\prod_{j \in I \cap A_0} \hat{s}_{j\phi_I(j)} \Bigg)
\end{aligned} \tag{3.22}$$

where ϕ_I is defined as

$$\phi_I : \begin{cases} j \longmapsto N-n+1 & \text{for } I = \{j \neq N-n+1\} \\ \text{empty} & \text{for } I = \{N-n+1\}\,. \end{cases} \tag{3.23}$$

Next assume that (3.21) is true, and check the case of $q+1$;

$$\Bigg(\prod_{m=N-n+1}^{N-n+q} D_{N-n}^m \Bigg) D_{N-n}^{N-n+q+1} = \sum_{\substack{I \subset A_q \\ |I|=q}} \Bigg(\prod_{\substack{i \in I \\ k \in I^c}} \sigma_\mu(z_{ik}) \Bigg) T_I \Bigg(\prod_{j \in I \cap A_0} \hat{s}_{j\phi_I(j)} \Bigg)$$

$$\times\left(\sum_{l\in A_0}\sigma_{(q+1)\mu}(z_{lN-n+q+1})\left(\prod_{\substack{m\in A_0\\ m\neq l}}\sigma_\mu(z_{lm})\right)T_l\hat{s}_{lN-n+q+1}\right.$$
$$\left.+\left(\prod_{m\in A_0}\sigma_\mu(z_{N-n+q+1m})\right)T_{N-n+q+1}\right). \tag{3.24}$$

We compute two terms on the right-hand side separately;

• the first term,

$$\sum_{\substack{I\subset A_q\\ |I|=q}}\left(\prod_{\substack{i\in I\\ k\in I^c}}\sigma_\mu(z_{ik})\right)T_I\left(\prod_{j\in I\cap A_0}\hat{s}_{j\phi_I(j)}\right)\sum_{l\in A_0}\sigma_{(q+1)\mu}(z_{lN-n+q+1})\left(\prod_{\substack{m\in A_0\\ m\neq l}}\sigma_\mu(z_{lm})\right)T_l\hat{s}_{lN-n+q+1}$$
$$=\sum_{\substack{I\subset A_q\\ |I|=q}}\left(\prod_{\substack{i\in I\\ k\in I^c}}\sigma_\mu(z_{ik})\right)\sum_{l\in I^c}\sigma_{(q+1)\mu}(z_{lN-n+q+1})$$
$$\times\left(\prod_{\substack{m\in I^c\\ m\neq l}}\sigma_\mu(z_{lm})\right)T_{I\cup\{l\}}\hat{s}_{lN-n+q+1}\left(\prod_{j\in I\cap A_0}\hat{s}_{j\phi_I(j)}\right)$$
$$=\sum_{\substack{I\subset A_q\\ |I|=q+1}}\left(\prod_{\substack{i\in I\\ k\in I^c}}\sigma_\mu(z_{ik})\right)\sum_{l\in I}\left(\prod_{\substack{m\in I\\ m\neq l}}\sigma_\mu(z_{ml})\right)\sigma_{(q+1)\mu}(z_{lN-n+q+1})$$
$$\times T_I\left(\prod_{j\in I\cap A_0}\hat{s}_{j\phi_I(j)}\right)$$
$$=\sum_{\substack{I\subset A_q\\ |I|=q+1}}\left(\prod_{\substack{i\in I\\ k\in I^c}}\sigma_\mu(z_{ik})\sigma_\mu(z_{iN-n+q+1})\right)T_I\left(\prod_{j\in I\cap A_0}\hat{s}_{j\phi_I(j)}\right)$$
$$=\sum_{\substack{I\subset A_{q+1}\\ |I|=q+1\\ N-j+q+1\notin I}}\left(\prod_{\substack{i\in I\\ k\in I^c}}\sigma_\mu(z_{ik})\right)T_I\left(\prod_{j\in I\cap A_0}\hat{s}_{j\phi_I(j)}\right).$$

We remark that, if $I\cup\{l\}=I'\cup\{l'\}$, we have

$$\hat{s}_{lN-n+q+1}\left(\prod_{j\in I\cap A_0}\hat{s}_{j\phi_I(j)}\right)=\hat{s}_{l'N-n+q+1}\left(\prod_{j\in I'\cap A_0}\hat{s}_{j\phi_{I'}(j)}\right) \tag{3.25}$$

on the symmetric space in $z_1,\ldots,z_{N-n}$ and $z_{N-n+1},\ldots,z_{N-n+q+1}$. By this remark, we defined $\phi_{I\cup\{l\}}$ in the third equality for a fixed $l\in I^c$ as follows. If $l\in A_0$

$$\phi_{I\cup\{l\}}(j)=\begin{cases}\phi_I(j) & \text{for } j\neq l\\ N-n+q+1 & \text{for } j=l\end{cases} \tag{3.26}$$

if $l\notin A_0$, there exists $l=\phi_I(n)$ and

$$\phi_{I\cup\{l\}}(j)=\begin{cases}\phi_I(j) & \text{for } j\neq n\\ N-n+q+1 & \text{for } j=n.\end{cases} \tag{3.27}$$

• The second term,

$$\sum_{\substack{I\subset A_q\\ |I|=q}}\left(\prod_{\substack{i\in I\\ k\in I^c}}\sigma_\mu(z_{ik})\right)T_I\left(\prod_{j\in I\cap A_0}\hat{s}_{j\phi_I(j)}\right)\left(\prod_{m\in A_0}\sigma_\mu(z_{N-n+q+1m})\right)T_{N-n+q+1}$$

$$= \sum_{\substack{I \subset A_q \\ |I|=q}} \left(\prod_{\substack{i \in I \\ k \in I^c}} \sigma_\mu(z_{ik}) \right) \left(\prod_{m \in I^c} \sigma_\mu(z_{N-n+q+1m}) \right) T_{I \cup \{N-j+q+1\}} \left(\prod_{j \in I \cap A_0} \hat{s}_{j\phi_I(j)} \right)$$

$$= \sum_{\substack{I \subset A_{q+1} \\ |I|=q+1 \\ N-n+q+1 \in I}} \left(\prod_{\substack{i \in I \\ k \in I^c}} \sigma_\mu(z_{ik}) \right) T_I \left(\prod_{j \in I \cap A_0} \hat{s}_{j\phi_I(j)} \right)$$

where in the last line, we define $\phi_{I \cup \{N-j+q+1\}}$ as

$$\phi_{I \cup \{N-j+q+1\}} : j \longmapsto \phi_I(j) \qquad \text{for } j \in I \cap A_0. \tag{3.28}$$

Combining these two terms, we obtain the final result (3.21) for $q+1$. Thus, by mathematical induction we conclude that (3.21) is satisfied for arbitrary q. □

The point is that the expansion is performed from the *left* of operators. By setting $q = n$ in lemma 3.5 and replacing all $\hat{s}_{jk}$ in the rightmost with identity operator, we complete the proof of theorem 3.3.

We note that we have determined only the difference of the spectral parameters ξ_{jk}. The condition $\xi_{jk} = (k-j)\mu$ is realized by setting $\xi_j = \alpha - j\mu$ for arbitrary constant α. One will see that this freedom plays the essential role in the D-type model.

To close this section, we show the duality of the Macdonald operators (3.6).

Theorem 3.6. The elliptic Macdonald operators have duality; the difference operators $\mathcal{M}_n(-\mu, -\beta)$ are given from $\mathcal{M}_m(\mu, \beta)$ as

$$\mathcal{M}_n(-\mu, -\beta) = (-1)^{(N-n)n} \mathcal{M}_{N-n}(\mu, \beta) (\mathcal{M}_N(\mu, \beta))^{-1} \qquad \text{for } 1 \leqslant n \leqslant N-1 \tag{3.29}$$

which shows that the difference operators $\mathcal{M}_n(-\mu, -\beta)$ commute with $\mathcal{M}_m(\mu, \beta)$.

Proof. This can be proved by a direct calculation. The right-hand side is computed as

$$\alpha_{N-n}\alpha_N \mathcal{M}_{N-n}(\mu, \beta)(\mathcal{M}_N(\mu, \beta))^{-1} = \left(\prod_{m=n+1}^{N} \left(\overleftarrow{\prod_{k=1}^{n}} R_\mu^{mk}(\xi_{mk}) \right) T_m(\beta) \right) \left(\prod_{l=1}^{N} T_l(-\beta) \right)$$

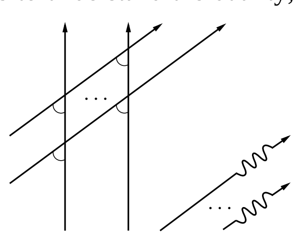
$$= \left(\overleftarrow{\prod_{m=1}^{N-n}} \prod_{k=N-n+1}^{N} R_\mu^{mk}(\xi_{km}) \right) \left(\prod_{l=N-n+1}^{N} T_l(-\beta) \right)$$

$$= \left(\prod_{k=N-n+1}^{N} \overleftarrow{\prod_{m=1}^{N-n}} (-1) R_{-\mu}^{km}(\xi_{mk}) \right) \left(\prod_{l=N-n+1}^{N} T_l(-\beta) \right)$$

$$= (-1)^{(N-n)n} \alpha_n \mathcal{M}_n(-\mu, -\beta)$$

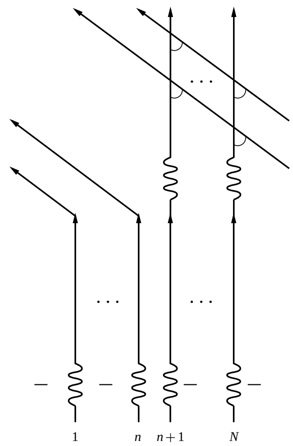
where we have exchanged the indices j and $N-j+1$ since $\mathcal{M}_n$ is a symmetric operator in $z_1, \ldots, z_N$. We have also used the fact that $\xi_{jk} = \xi_{lm}$ if $j-k = l-m$. Observing $\alpha_n = \alpha_{N-n}$ and $\alpha_N = 1$, we obtain result (3.29).

Figures help us to understand the duality;

${}_n\mathcal{M}_n . \; ; \; / \; =$ 

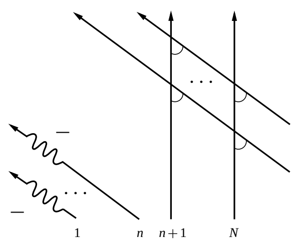
(3.30a)

$$ {}_{N-n}\ {}_{n}\mathcal{M}_{N-n}\,.\ ;\ /\left(\mathcal{M}_N\,.\ ;\ /\right)^{-1} \quad = $$

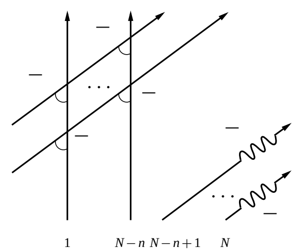


(3.30*b*)

$=$



$=$



$\square$

4. Elliptic Ruijsenaars model of type D

In this section, we propose a new integrable relativistic Hamiltonian system of type D (1.2). We study a set of the D-type Ruijsenaars operators $\mathcal{W}_n$; the lowest-order operator is given as

$$\begin{aligned}
\mathcal{W}_1 = {} & \sum_{j=1}^{N}\left(\prod_{\substack{k=1\\k\neq j}}^{N}\sigma_\mu(z_{jk})\sigma_\mu(z_j+z_k)\right)\left(\sum_{r=0}^{3} g_r\sigma_{2\nu}^{r}(z_j)\right)\\
& \times\left(\sum_{r=0}^{3}\bar{g}_r\sigma_{2\bar{\nu}}^{r}(z_j+\beta)T_j^2(\beta)-\sum_{r=0}^{3}\bar{g}_r\sigma^{r}_{-2(N-1)\mu-2\nu}(z_j+\beta)\right)\\
& +\sum_{j=1}^{N}\left(\prod_{\substack{k=1\\k\neq j}}^{N}\sigma_\mu(z_{kj})\sigma_\mu(-z_j-z_k)\right)\left(\sum_{r=0}^{3} g_r\sigma_{2\nu}^{r}(-z_j)\right)\\
& \times\left(\sum_{r=0}^{3}\bar{g}_r\sigma_{2\bar{\nu}}^{r}(-z_j+\beta)T_j^{-2}(\beta)-\sum_{r=0}^{3}\bar{g}_r\sigma^{r}_{-2(N-1)\mu-2\nu}(-z_j+\beta)\right).
\end{aligned}\tag{4.1}$$

One sees that the operator is invariant under an exchange $z_j \leftrightarrow z_k$ and a reflection $z_j \leftrightarrow -z_j$. We impose such a restriction on the space in the following. It will be shown later that the difference operator $\mathcal{W}_1$ includes a gauge-transformed operator of the Hamiltonian $\mathcal{H}_{\mathrm{D}}$ (1.2). We shall prove that the model (4.1) is integrable, and that the higher-order difference operators $\mathcal{W}_n$ can be defined in terms of our elliptic R-operator (2.9) and K-operator (2.10). We will also clarify the relation between the operator (4.1) and the model proposed in [2].

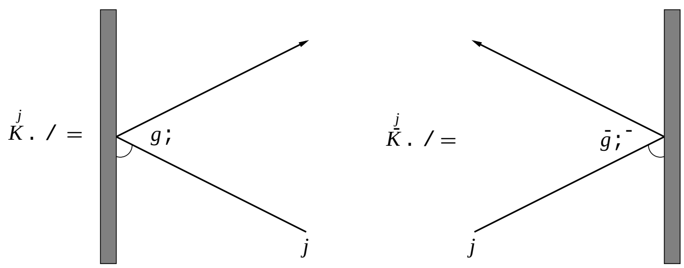
First we define a set of difference operators $\mathcal{Y}_n(\xi)$ for $\xi = \{\xi_1, \ldots, \xi_N\}$ as

$$\mathcal{Y}_n(\xi) = \left(\prod_{m=N-n+1}^{N} \overleftarrow{\prod_{k=1}^{N-n}} R^{mk}(\xi_{mk}) \right) \times \left(\prod_{k=N-n+1}^{N} \overset{k}{K}(\xi_k) \left(\prod_{l=k+1}^{N} R^{lk}(\xi_l + \xi_k) \right) \left(\prod_{l=1}^{k-1} R^{lk}(\xi_l + \xi_k) \right) \overset{k}{\bar{K}}(\xi_k) \right). \quad (4.2)$$

In this expression, we have used a 'conjugate' boundary operator $\bar{K}(\xi)$ as

$$\bar{K}(\xi) = T(-\beta)\hat{t}K(\xi)\hat{t}T(\beta) \quad (4.3)$$

where parameters in the elliptic K-operator (2.10) are replaced by $g_r \to \bar{g}_r$ and $\nu \to \bar{\nu}$. The operators K and $\bar{K}$ are respectively depicted as follows



(4.4)

As the boundary K-operator (2.10) satisfies the RE (2.2), we have the following proposition.

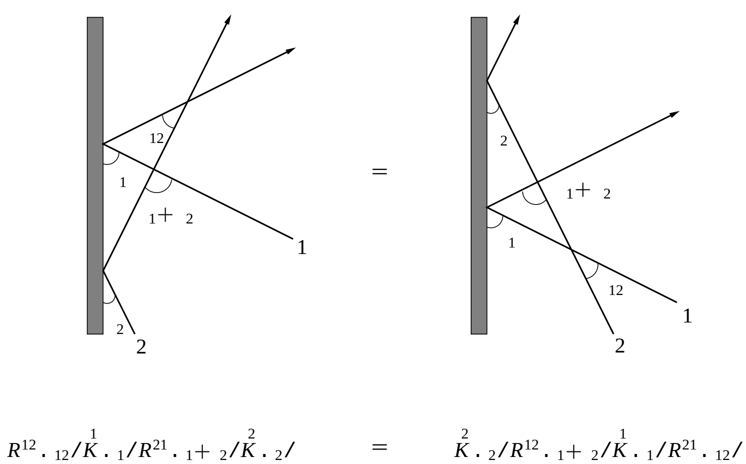
Proposition 4.1. The boundary operator $\bar{K}(\xi)$ defined in (4.3) satisfies the 'conjugate' reflection equation;

$$\overset{2}{\bar{K}}(\xi_2) R^{21}(\xi_1 + \xi_2) \overset{1}{\bar{K}}(\xi_1) R^{12}(\xi_{12}) = R^{21}(\xi_{12}) \overset{1}{\bar{K}}(\xi_1) R^{12}(\xi_1 + \xi_2) \overset{2}{\bar{K}}(\xi_2). \quad (4.5)$$

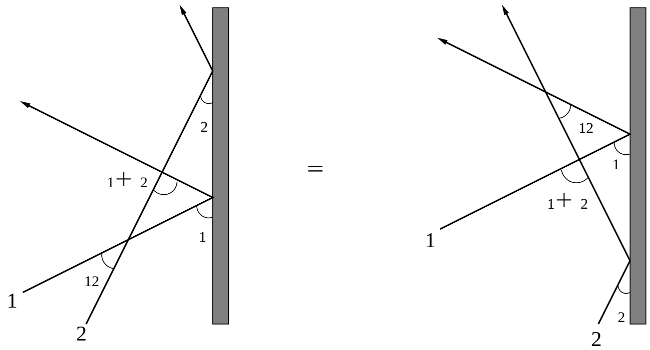
Here $R(\xi)$ denotes the elliptic R-operator (2.9).

Proof. In the RE (2.2), multiplying $T_1(-\beta)T_2(-\beta)\hat{t}_1\hat{t}_2$ from the left and $\hat{t}_1\hat{t}_2T_1(\beta)T_2(\beta)$ from the right, we obtain the conjugate reflection equation (4.5). □

The reflection equation (2.2) and the conjugate RE (4.5) can be graphically interpreted as follows



(2.2′)

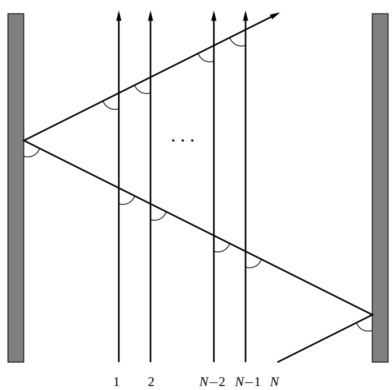


(4.5′)

$$\overset{2}{\tilde{K}}(\xi_2)R^{21}(\xi_1+\xi_2)\overset{1}{\tilde{K}}(\xi_1)R^{12}(\xi_{12}) = R^{21}(\xi_{12})\overset{1}{\tilde{K}}(\xi_1)R^{12}(\xi_1+\xi_2)\overset{2}{\tilde{K}}(\xi_2).$$

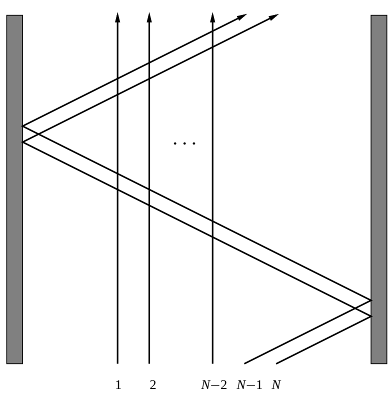
Using the above interpretations, the difference operators $\mathcal{Y}_n(\xi)$ defined in (4.2) can be depicted as follows

$\mathcal{Y}_1(\xi) =$



(4.6*a*)

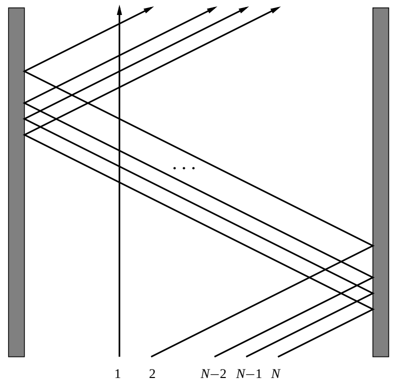
$\mathcal{Y}_2(\xi) =$



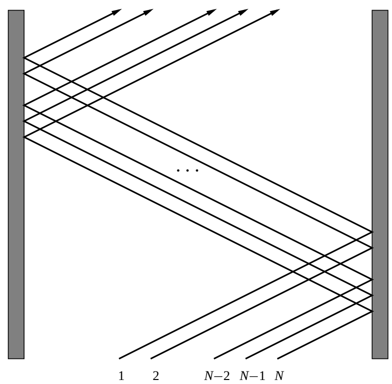
(4.6*b*)

$\vdots$

$$\mathcal{Y}_{N-1}\,.\;/ = \quad \text{(4.6c)}$$



$$\mathcal{Y}_{N}\,.\;/= \quad \text{(4.6d)}$$



For these operators we have the following theorem, which shows the quantum integrability.

Theorem 4.2. The difference operators (4.2) are integrable,

$$[\mathcal{Y}_l(\xi), \mathcal{Y}_m(\xi)] = 0 \qquad \text{for } 1 \leqslant l,\ \ m \leqslant N. \tag{4.7}$$

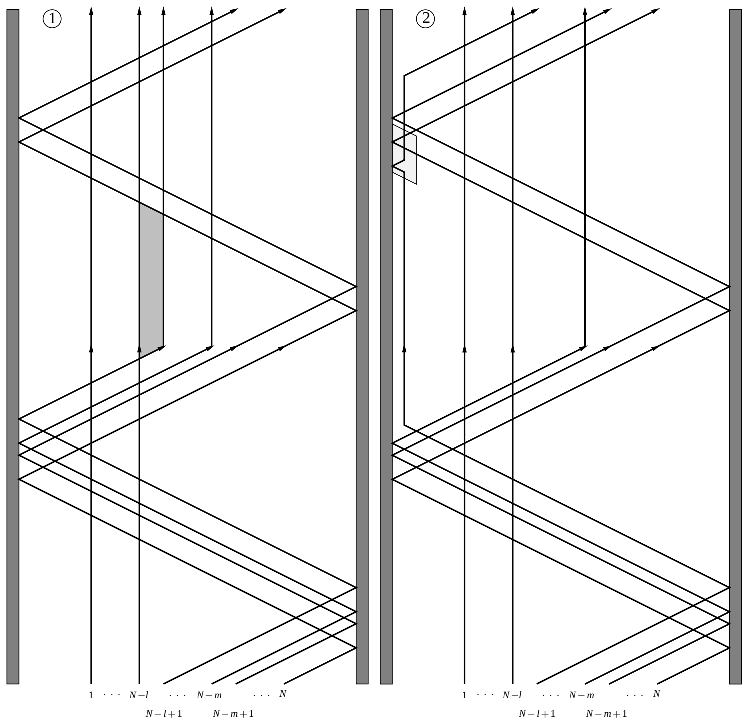
Proof. Similar to the proof of the A-type Macdonald operators (theorem 3.2), we can prove this statement by the extended railway argument.

(1) We depict the product $\mathcal{Y}_l(\xi)\mathcal{Y}_m(\xi)$ for $l > m$ (the left of (4.8*a*)).

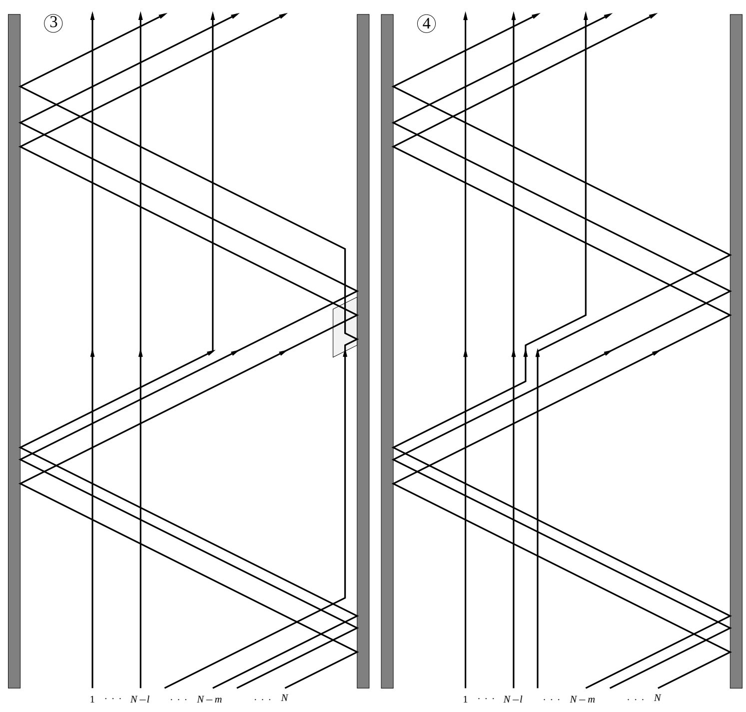
(2) By repeatedly using the YBE (2.1) and the RE (2.2), (4.5), we can move the $(N - l)$th line upwards until the upper half and the lower half is exchanged (the right of (4.8*a*) and (4.8*b*)).

(3) Successively applying the above steps for $N - l + 1, \ldots, N - m$, we arrive at (4.8*c*),

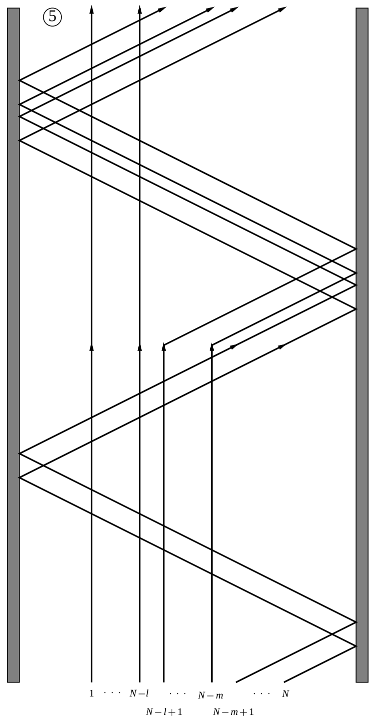
which implies the product $\mathcal{Y}_m(\xi)\mathcal{Y}_l(\xi)$.



(4.8*a*)



(4.8*b*)



(4.8*c*)

In conclusion, we obtain an equality, $\mathcal{Y}_l(\xi)\mathcal{Y}_m(\xi) = \mathcal{Y}_m(\xi)\mathcal{Y}_l(\xi)$. □

Next we show that the first-order operator $\mathcal{W}_1$ (4.1) agrees with $\mathcal{Y}_1(\xi)$ (4.2) when spectral parameters $\{\xi_j\}$ are fixed adequately.

Theorem 4.3. We set the spectral parameters as $\xi_k = -\nu-(k-1)\mu$ in the difference operator $\mathcal{Y}_1(\xi)$ (4.2). The elliptic Macdonald–Koornwinder operator (4.1) is then given as

$$\mathcal{W}_1 = \mathcal{Y}_1. \tag{4.9}$$

Here we suppose that the functional space is symmetric under $\hat{s}_{jk}$ and $\hat{t}_j$.

Proof. To prove this theorem, we use the following identity which is derived using lemma A.2.

Lemma 4.4. The elliptic R-operator (2.9) satisfies the formula,

$$\prod_{m=1}^{N-1} R^{mN}(\xi_m + \xi_N) = \prod_{m=1}^{N-1} \sigma_\mu(z_{mN}) - \sum_{l=1}^{N-1} \sigma_{\xi_1+\xi_N}(z_{lN})\Bigg(\prod_{\substack{m=1\\ m\neq l}}^{N-1} \sigma_\mu(z_{ml})\Bigg)\hat{s}_{lN}. \tag{4.10}$$

We first expand the operator $\mathcal{Y}_1$ by using lemmas 4.4 and A.2. When we substitute the elliptic R-operator (2.9), we obtain,

$$\mathcal{Y}_1 = \sum_{j=1}^{N}\Bigg(\prod_{\substack{k=1\\ k\neq j}}^{N} \sigma_\mu(z_{jk})\Bigg)\overset{j}{K}(\xi_N)\hat{s}_{jN}$$
$$\times\Bigg(\Bigg(\prod_{m=1}^{N-1}\sigma_\mu(z_{mN})\Bigg)\overset{N}{\bar{K}}(\xi_N) - \sum_{l=1}^{N-1}\sigma_{\xi_1+\xi_N}(z_{lN})\Bigg(\prod_{\substack{m=1\\ m\neq l}}^{N-1}\sigma_\mu(z_{ml})\Bigg)\overset{l}{\bar{K}}(\xi_N)\Bigg)$$

$$= -\sum_{j=1}^{N}\left(\prod_{\substack{k=1\\k\neq j}}^{N}\sigma_\mu(z_{jk})\right)H(z_j)\left(\prod_{\substack{m=1\\m\neq j}}^{N}\sigma_\mu(z_m+z_j)\right)\overset{j}{\check{\bar{K}}}\ (\xi_N) \tag{4.11a}$$

$$+\sum_{j=1}^{N}\left(\prod_{\substack{k=1\\k\neq j}}^{N}\sigma_\mu(z_{jk})\right)G(\xi_N,z_j)\left(\prod_{\substack{m=1\\m\neq j}}^{N}\sigma_\mu(z_{mj})\right)\overset{j}{\bar{K}}\ (\xi_N) \tag{4.11b}$$

$$+\sum_{j=1}^{N}\left(\prod_{\substack{k=1\\k\neq j}}^{N}\sigma_\mu(z_{jk})\right)H(z_j)\sum_{\substack{l=1\\l\neq j}}^{N}\sigma_{\xi_1+\xi_N}(z_l+z_j)\left(\prod_{\substack{m=1\\m\neq j,l}}^{N}\sigma_\mu(z_{ml})\right)\overset{l}{\bar{K}}\ (\xi_N) \tag{4.11c}$$

$$-\sum_{j=1}^{N}\left(\prod_{\substack{k=1\\k\neq j}}^{N}\sigma_\mu(z_{jk})\right)G(\xi_N,z_j)\sum_{\substack{l=1\\l\neq j}}^{N}\sigma_{\xi_1+\xi_N}(z_{lj})\left(\prod_{\substack{m=1\\m\neq j,l}}^{N}\sigma_\mu(z_{ml})\right)\overset{l}{\bar{K}}\ (\xi_N) \tag{4.11d}$$

where $\overset{j}{\check{\bar{K}}}\ (\xi_N)$ denotes $\overset{j}{\bar{K}}\ (\xi_N)$ replacing $z_j \to -z_j$. Recall the definition of the boundary K-operator in (2.4) and (2.10) as for functions $G(\xi, z)$ and $H(z)$. One sees that the first term (4.11a) is identical to the first term of (4.1). Thus, we only have to calculate the last three terms (4.11b)–(4.11d).

As we know that $G(\nu, z_j) = H(z_j)$ and $\xi_1 = -\nu$, we have

$$\begin{aligned}(4.11b)+(4.11c)+(4.11d) &= \sum_{j=1}^{N}\prod_{\substack{k=1\\k\neq j}}^{N}\sigma_\mu(z_{kj})\Bigg(\Bigg(\prod_{\substack{m=1\\m\neq j}}^{N}\sigma_\mu(z_{jm})\Bigg)G(\xi_N,z_j)\\ &\quad -\sum_{\substack{l=1\\l\neq j}}^{N}\prod_{\substack{m=1\\m\neq j,l}}^{N}\sigma_\mu(z_{lm})(\sigma_{\xi_1+\xi_N}(z_j+z_l)G(\xi_1,-z_l)\\ &\quad +\sigma_{\xi_1+\xi_N}(z_{jl})G(\xi_N,z_l))\Bigg)\overset{j}{\bar{K}}\ (\xi_N).\end{aligned} \tag{4.12}$$

To simplify this form, we use the fact that $G(\xi, z)$ is the solution of the functional equation (2.8a) with $B(\xi, z) = \sigma_\xi(z)$,

$$\begin{aligned}&\sigma_{\xi_1+\xi_N}(z_j+z_l)G(\xi_1,-z_l)+\sigma_{\xi_1+\xi_N}(z_{jl})G(\xi_N,z_l)\\ &\qquad = \sigma_{\xi_{N1}}(z_j+z_l)G(\xi_1,z_j)-\sigma_{\xi_{N1}}(z_{lj})G(\xi_N,z_j).\end{aligned}$$

We also use the identities;

$$-\sum_{\substack{l=1\\l\neq j}}^{N}\left(\prod_{\substack{m=1,\\m\neq j,l}}^{N}\sigma_\mu(z_{lm})\right)\sigma_{\xi_{N1}}(z_j+z_l)G(\xi_1,z_j) = -\left(\prod_{\substack{m=1\\m\neq j}}^{N}\sigma_\mu(-z_m-z_j)\right)H(-z_j) \tag{4.13}$$

$$\prod_{\substack{m=1\\m\neq j}}^{N}\sigma_\mu(z_{jm})+\sum_{\substack{l=1\\l\neq j}}^{N}\left(\prod_{\substack{m=1\\m\neq j,l}}^{N}\sigma_\mu(z_{lm})\right)\sigma_{\xi_{N1}}(z_{lj}) = 0 \tag{4.14}$$

which come from lemma A.2 and the fact $G(\nu, z_j) = H(z_j)$. Using these properties, we find that (4.12) reduces to the second term of (4.1). Thus, we obtain the explicit form of the difference operator $\mathcal{Y}_1$ as

$$\mathcal{Y}_1 = -\sum_{j=1}^{N}\left(\prod_{\substack{k=1\\k\neq j}}^{N}\sigma_\mu(z_{jk})\right)H(z_j)\left(\prod_{\substack{m=1\\m\neq j}}^{N}\sigma_\mu(z_m+z_j)\right)\overset{j}{\check{\bar{K}}}(\xi_N)$$
$$-\sum_{j=1}^{N}\left(\prod_{\substack{k=1\\k\neq j}}^{N}\sigma_\mu(z_{kj})\right)H(-z_j)\left(\prod_{\substack{m=1\\m\neq j}}^{N}\sigma_\mu(-z_m-z_j)\right)\overset{j}{\bar{K}}(\xi_N) \tag{4.15}$$

which coincides with the operator $\mathcal{W}_1$ (4.1). □

The generalized elliptic Ruijsenaars model was studied in [2], where the Hamiltonian contained nine arbitrary parameters. To see the relationship with our model (4.1), we use the following lemma.

Lemma 4.5. For a given set of parameters ν_r, ($r = 0, 1, 2, 3$), the following identity holds;

$$\sum_{r=0}^{3} g_r\sigma_{2\nu}^{r}(z) = \prod_{r=0}^{3}\sigma_{\nu_r}^{r}(z) \tag{4.16}$$

if we set $g_r = \prod_{\substack{s=0\\s\neq r}}^{3}\sigma_{\nu_s}^{\pi_r s}(0)$ and $2\nu = \sum_{r=0}^{3}\nu_r$. Using elementary transpositions of $\mathfrak{S}^4$, permutations π_r are defined as $\pi_0 = \mathbb{1}$, $\pi_1 = (01)(23)$, $\pi_2 = (02)(13)$, and $\pi_3 = (03)(12)$.

In the same manner, when we set $\bar{g}_r = \prod_{\substack{s=0\\s\neq r}}^{3}\sigma_{\bar{\nu}_s}^{\pi_r s}(0)$ and $2\bar{\nu} = \sum_{r=0}^{3}\bar{\nu}_r$ for a given set $\bar{\nu}_r$, we have

$$\sum_{r=0}^{3}\bar{g}_r\sigma_{2\bar{\nu}}^{r}(z) = \prod_{r=0}^{3}\sigma_{\bar{\nu}_r}^{r}(z). \tag{4.17}$$

Using these properties, we have the following lemma.

Lemma 4.6. Under the condition in lemma 4.5, we have an identity,

$$\sum_{j=1}^{N}\left(\prod_{\substack{k=1\\k\neq j}}^{N}\sigma_\mu(z_{jk})\sigma_\mu(z_j+z_k)\right)\left(\sum_{r=0}^{3}g_r\sigma_{2\nu}^{r}(z_j)\right)\left(\sum_{r=0}^{3}\bar{g}_r\sigma_{-2(N-1)\mu-2\nu}^{r}(z_j+\beta)\right)$$
$$+\sum_{j=1}^{N}\left(\prod_{\substack{k=1\\k\neq j}}^{N}\sigma_\mu(z_{kj})\sigma_\mu(-z_j-z_k)\right)\left(\sum_{r=0}^{3}g_r\sigma_{2\nu}^{r}(-z_j)\right)$$
$$\times\left(\sum_{r=0}^{3}\bar{g}_r\sigma_{-2(N-1)\mu-2\nu}^{r}(-z_j+\beta)\right)$$
$$=\sum_{p=0}^{3}\frac{1}{\sigma_\mu(-2\beta)}\left(\prod_{r=0}^{3}\sigma_{\nu_r}^{\pi_p r}(-\beta)\right)\left(\prod_{\substack{s=0\\s\neq p}}^{3}\sigma_{\bar{\nu}_s}^{\pi_p s}(0)\right)$$
$$\times\left(\prod_{j=1}^{N}\sigma_\mu^{p}(z_j-\beta)\sigma_\mu^{p}(-z_j-\beta)\right). \tag{4.18}$$

Proof. Both sides are functions of z_j with double periodicity. Comparing poles, we conclude that the difference is a constant. The quasiperiodicity in β implies the constant is zero, which proves identity (4.18). □

We now rewrite the difference operator $\mathcal{W}_1$ (4.1). In the following we use the identities;

$$2\prod_{r=0}^{3}\vartheta_r(z)=\vartheta_1(2z)\vartheta_2(0)\vartheta_3(0)\vartheta_0(0) \tag{4.19}$$

$$\vartheta_1'(0)=\pi\vartheta_2(0)\vartheta_3(0)\vartheta_0(0). \tag{4.20}$$

After some computations, we obtain an expression for arbitrary ν_r and $\bar{\nu}_r$,

$$\begin{aligned}\left(\frac{\vartheta_1(-\mu)}{\vartheta_1'(0)}\right)^{2(N-1)}\prod_{r=0}^{3}\left(\frac{\vartheta_1(-\nu_r)\vartheta_1(-\bar{\nu}_r)}{\vartheta_1'(0)^2}\right)\mathcal{W}_1&=\sum_{j=1}^{N}\left(\prod_{\substack{k=1\\k\neq j}}^{N}\frac{\vartheta_1(z_{jk}-\mu)}{\vartheta_1(z_{jk})}\frac{\vartheta_1(z_j+z_k-\mu)}{\vartheta_1(z_j+z_k)}\right)\\&\times\left(\prod_{r=0}^{3}\frac{\vartheta_{r+1}(z_j-\nu_r)}{\vartheta_{r+1}(z_j)}\frac{\vartheta_{r+1}(z_j+\beta-\bar{\nu}_r)}{\vartheta_{r+1}(z_j+\beta)}\right)T_j^2(\beta)\\&+\sum_{j=1}^{N}\left(\prod_{\substack{k=1\\k\neq j}}^{N}\frac{\vartheta_1(z_{kj}-\mu)}{\vartheta_1(z_{kj})}\frac{\vartheta_1(-z_j-z_k-\mu)}{\vartheta_1(-z_j-z_k)}\right)\\&\times\left(\prod_{r=0}^{3}\frac{\vartheta_{r+1}(-z_j-\nu_r)}{\vartheta_{r+1}(-z_j)}\frac{\vartheta_{r+1}(-z_j+\beta-\bar{\nu}_r)}{\vartheta_{r+1}(-z_j+\beta)}\right)T_j^{-2}(\beta)\\&-\sum_{p=0}^{3}\left(\frac{\pi}{\vartheta_1'(0)}\right)^2\frac{2}{\vartheta_1(-\mu)\vartheta_1(-2\beta-\mu)}\left(\prod_{r=0}^{3}\vartheta_{r+1}(-\beta-\nu_{\pi_p r})\vartheta_{r+1}(-\bar{\nu}_{\pi_p r})\right)\\&\times\left(\prod_{j=1}^{N}\frac{\vartheta_{p+1}(z_j-\beta-\mu)}{\vartheta_{p+1}(z_j-\beta)}\frac{\vartheta_{p+1}(-z_j-\beta-\mu)}{\vartheta_{p+1}(-z_j-\beta)}\right).\end{aligned} \tag{4.21}$$

This operator coincides with one studied in [2], where the constraint $\sum_{r=0}^{3}(\nu_r+\bar{\nu}_r)=0$ was conjectured.

It is remarked that the operator (4.21) is connected to the D-type Hamiltonian (1.2) by a gauge transformation,

$$\mathcal{W}_1\equiv\Delta_{\mathrm{D}}^{-1/2}\mathcal{H}_{\mathrm{D}}\Delta_{\mathrm{D}}^{1/2}. \tag{4.22}$$

The function Δ_{D} is given by

$$\Delta_{\mathrm{D}}=\left(\prod_{1\leqslant j<k\leqslant N}c_V(z_j+z_k)c_V(-z_j-z_k)c_V(z_{jk})c_V(z_{kj})\right)\left(\prod_{1\leqslant j\leqslant N}c_W(z_j)c_W(-z_j)\right) \tag{4.23a}$$

$$c_V(z)=\frac{(pv;p,q^2)_\infty}{(pvw^{-1};p,q^2)_\infty}\frac{(q^2v^{-1}w;p,q^2)_\infty}{(q^2v^{-1};p,q^2)_\infty} \tag{4.23b}$$

$$\begin{aligned}c_W(z)=&\left(\prod_{r=0}^{3}\frac{((-1)^{a_r}p^{1-b_r/2}v;p,q^2)_\infty}{((-1)^{a_r}p^{1-b_r/2}vy_r^{-1};p,q^2)_\infty}\frac{((-1)^{a_r}p^{b_r/2}q^2v^{-1}y_r;p,q^2)_\infty}{((-1)^{a_r}p^{b_r/2}q^2v^{-1};p,q^2)_\infty}\right)\\&\times\left(\prod_{r=0}^{3}\frac{((-1)^{a_r}p^{1-b_r/2}qv;p,q^2)_\infty}{((-1)^{a_r}p^{1-b_r/2}qv\bar{y}_r^{-1};p,q^2)_\infty}\frac{((-1)^{a_r}p^{b_r/2}qv^{-1}\bar{y}_r;p,q^2)_\infty}{((-1)^{a_r}p^{b_r/2}qv^{-1};p,q^2)_\infty}\right).\end{aligned} \tag{4.23c}$$

We set parameters as $v=\mathrm{e}^{2\pi\mathrm{i}z}$, $w=\mathrm{e}^{2\pi\mathrm{i}\mu}$, $p=\mathrm{e}^{2\pi\mathrm{i}\tau}$, $q=\mathrm{e}^{2\pi\mathrm{i}\beta}$, $y_r=\mathrm{e}^{2\pi\mathrm{i}\nu_r}$, and $\bar{y}_r=\mathrm{e}^{2\pi\mathrm{i}\bar{\nu}_r}$. See (A.3) for definitions of a_r and b_r. From this observation, we can regard operator (4.1) as a generalization of the D-type elliptic Ruijsenaars model.

5. Concluding remarks

In this paper, we have proposed a new construction of difference operators acting on the symmetric functional spaces, which can be considered as elliptic generalizations of the Macdonald operators and the Macdonald–Koornwinder operators. Pictorial interpretations of the operators naturally result in the integrability of the difference operators. We have shown that these models coincide with the gauge-transformed operators of the elliptic Ruijsenaars model and its D-type analogue.

Our technical tools are operator-valued solutions of the YBE and the RE, i.e. the solutions acting on the functional spaces. Although these solutions are also used in the previously known construction, the way we used them in this paper is quite different and is applicable even for the elliptic Ruijsenaars models. In addition, the D-type operator (4.1) is shown to be a generalization of the model proposed in [2]. Note that if we replace the boundary operator $K(\xi)$ by $T(-\beta')K(\xi)T(\beta')$, this operator still satisfies the RE (2.2) and thus we obtain a generalization with one more parameter. As for the A-type model, we have calculated the explicit forms of all the conserved operators and have shown that they coincide with the higher-order Macdonald operators.

Finally we comment on the open problems. Some of them are;

- the algebra underlying these elliptic models,
- the explicit forms of the D-type higher-order conserved operators.

For the latter problem, the explicit forms were suggested in [2], and we believe this conjecture should be dealt with in our way.

Acknowledgments

The authors would like to thank Miki Wadati for his kind interests in this work. One of the authors (KH) also thanks P J Forrester for stimulating discussions and kind hospitality at University of Melbourne. This work was supported in part by the Grant-in-Aid from the Ministry of Education, Science, Sports and Culture, of Japan.

Appendix. Fundamental functions and identities

We shall establish notations and useful identities on the theta functions [22, 23] used in this paper.

The Jacobi theta functions are defined for $\Im\tau > 0$ as,

$$\vartheta_1(z) = -\mathrm{i}\sum_{n\in\mathbb{Z}} \exp(\mathrm{i}\pi(n+\tfrac{1}{2})^2\tau + 2\pi\mathrm{i}(n+\tfrac{1}{2})z + \mathrm{i}\pi n) \tag{A.1a}$$

$$\vartheta_2(z) = \sum_{n\in\mathbb{Z}} \exp(\mathrm{i}\pi(n+\tfrac{1}{2})^2\tau + 2\pi\mathrm{i}(n+\tfrac{1}{2})z) \tag{A.1b}$$

$$\vartheta_3(z) = \sum_{n\in\mathbb{Z}} \exp(\mathrm{i}\pi n^2\tau + 2\pi\mathrm{i}nz) \tag{A.1c}$$

$$\vartheta_0(z) = \sum_{n\in\mathbb{Z}} \exp(\mathrm{i}\pi n^2\tau + 2\pi\mathrm{i}nz + \mathrm{i}\pi n) \tag{A.1d}$$

and $\vartheta_4(z) = \vartheta_0(z)$. These functions have a kind of periodic behaviour under the translation generated by 1 and τ. Note that $\vartheta_1(z)$ is odd while the other three are even.

Let us define functions $\sigma_\mu^r(z)$ for $r = 0, 1, 2, 3$ in terms of the Jacobi theta functions by

$$\sigma_\mu^r(z) = \frac{\vartheta_{r+1}(z-\mu)\vartheta_1'(0)}{\vartheta_{r+1}(z)\vartheta_1(-\mu)} \tag{A.2}$$

where $\vartheta_1'(z)$ denotes a derivative of $\vartheta_1(z)$ with respect to z. In this paper we often refer to $\sigma_\mu^0(z)$ as $\sigma_\mu(z)$ for brevity. The following properties uniquely characterize the functions $\sigma_\mu^r(z)$;

- meromorphic in μ with simple poles at $\mathbb{Z} + \tau\mathbb{Z}$;
- meromorphic in z with simple poles at $\mathbb{Z} + \tau\mathbb{Z} + \omega_r$ ($r = 0, 1, 2, 3$). The residue at $z = \omega_r$ is $e^{\pi i \mu b_r}$, where each parameter is defined respectively as

r	$\omega_r = (a_r + b_r\tau)/2$	a_r	b_r
0	0	0	0
1	$\frac{1}{2}$	1	0
2	$\frac{1}{2} + \tau/2$	1	1
3	$\tau/2$	0	1

(A.3)

- doubly quasiperiodic,

$$\sigma_\mu^r(z+1) = \sigma_\mu^r(z) \qquad \sigma_\mu^r(z+\tau) = e^{2\pi i \mu}\sigma_\mu^r(z). \tag{A.4}$$

We remark that the elliptic functions $\sigma_\mu^r(z)$ satisfy the formulae [22, 23],

$$\sigma_{-\mu}^r(-z) = -\sigma_\mu^r(z) \tag{A.5}$$

$$\sigma_\lambda(z)\sigma_\mu(w) = \sigma_{\lambda+\mu}(w)\sigma_\lambda(z-w) + \sigma_\mu(w-z)\sigma_{\lambda+\mu}(z) \tag{A.6}$$

$$\sigma_\mu^r(z)\sigma_\mu^r(-z) = \wp(\mu) - \wp(z+\omega_r) \tag{A.7}$$

where $\wp(z)$ is Weierstrass' $\wp$-function with periods 1 and τ.

To close this section, we give two lemmas.

Lemma A.1. If a function $K_\mu(z)$ satisfies the following conditions;

(1) $K_\mu(z)$ is meromorphic in μ, and a set of possible poles Ω is independent of z;

(2) $K_\mu(z)$ is entire in z, where $\mu \in D = \mathbb{C}\backslash\Omega$;

(3) $K_\mu(z)$ is doubly quasiperiodic, $K_\mu(z+1) = K_\mu(z)$ and $K_\mu(z+\tau) = e^{2\pi i\mu}K_\mu(z)$,

then $K_\mu(z)$ is identically zero.

Proof. It can be proved by using the Liouville theorem. □

Lemma A.2. The elliptic function $\sigma_\mu(z)$ satisfies the generalized addition formula for arbitrary z_j and μ_j;

$$\sum_{i=1}^{q}\left(\prod_{\substack{k=1\\k\neq i}}^{q}\sigma_{\mu_k}(z_{ki})\right)\sigma_{\sum_{j=1}^{q}\mu_j}(z_i) = \prod_{k=1}^{q}\sigma_{\mu_k}(z_k). \tag{A.8}$$

Proof. This is proved by using lemma A.1. □

Note that lemma A.2 with $q = 2$ coincides with the addition formula (A.6). We use this identity only in the case $\mu_j = \mu$ for all j.

References

[1] Ruijsenaars S N M 1987 *Commun. Math. Phys.* **110** 191
[2] van Diejen J F 1994 *J. Math. Phys.* **35** 2983
[3] van Diejen J F 1995 *J. Math. Phys.* **36** 1299
[4] Macdonald I G 1994–95 *Séminaire Bourbaki* **797** 1
[5] Macdonald I G 1995 *Symmetric Functions and Hall Polynomials* 2nd edn (Oxford: Oxford University Press)
[6] Koornwinder T H 1992 *Contemp. Math.* **138** 189
[7] Gaudin M 1988 *J. Physique* **49** 1857
[8] Shibukawa Y and Ueno K 1992 *Lett. Math. Phys.* **25** 239
[9] Hikami K 1995 *Phys. Lett.* A **205** 167

[10] Komori Y and Hikami K *Nucl. Phys.* B to appear

[11] Noumi M 1995 *RIMS Kôkyûroku* **919** 44 (in Japanese)

[12] Hikami K 1996 *J. Phys. A: Math. Gen.* **29** L281

[13] Hikami K and Komori Y 1996 *Mod. Phys. Lett.* A **11** 2861

[14] Buchstaber V M, Felder G and Veselov A P 1994 *Duke Math. J.* **76** 885

[15] Hikami K 1996 *J. Phys. A: Math. Gen.* **29** 2135

[16] Hikami K and Wadati M 1996 *Preprint* to appear

[17] Hasegawa K 1995 *Preprint* q-alg/9512029

[18] Nijhoff F W, Kuznetsov V B, Sklyanin E K and Ragnisco O 1996 *J. Phys. A: Math. Gen.* **29** L333

[19] Kuznetsov V B, Nijhoff F W and Sklyanin E K 1997 *Preprint* solv-int/9701004

[20] Cherednik I 1984 *Theor. Math. Phys.* **61** 35

[21] Sklyanin E K 1988 *J. Phys. A: Math. Gen.* **21** 2375

[22] Whittaker E T and Watson G N 1927 *A Course of Modern Analysis* 4th edn (Cambridge: Cambridge University Press)

[23] Hurwitz A and Courant R 1964 *Vorlesungen über Allgemeine Funktionentheorie und Elliptische Funktionen*, (Berlin: Springer)